AP CALCULUS AB
DR. PAUL L. BAILEY

Homework 4.2a
Tuesday, December 17, 2019

Name:

Write your homework *neatly, in pencil,* on blank white $8\frac{1}{2} \times 11$ printer paper. Always *write the problem,* or at least enough of it so that your work is readable. When appropriate, *write in sentences.*

The Mean Value Theorem is stated below.

Theorem 1. (Rolle's Theorem)
Let f be continuous on a closed interval $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = f(b) = 0$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Theorem 2. (Mean Value Theorem (MVT))
Let f be continuous on a closed interval $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 1 (Thomas §4.2 # 4)**.** Let $f(x) = \sqrt{x-1}$. Let $a = 1$ and $b = 3$. Find $c \in [a,b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 2 (Thomas §4.2 # 5 - 8)**.** Which functions satisfy the Mean Value Theorem on the indicated interval, and which do not? Justify you answer.

(a) $f(x) = x^{2/3}$ on $[-1, 8]$

(b) $f(x) = x^{4/5}$ on $[0, 1]$

(c) $f(x) = \sqrt{x(1-x)}$ on $[0, 1]$

(d) $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for } x \in [-\pi, 0) \\ 0 & \text{for } x = 0 \end{cases}$

Problem 3 (Thomas §4.2 # 10)**.** Let

$$f(x) = \begin{cases} 3 & \text{for } x = 0 \\ -x^2 + 3x + a & \text{for } x \in (0,1) \\ mx + b & \text{for } x \in [1,2] \end{cases}$$

For what values of a, m, and b does f satisfy the hypothesis of the Mean Value Theorem on the interval $[0, 2]$?

Problem 4 (Thomas §4.2 # 15)**.** Show that the function

$$f(x) = x^4 + 3x + 1$$

has exactly one zero on $[-2, -1]$.

Problem 5 (Thomas §4.2 # 19)**.** Show that the function

$$r(\theta) = \theta + \sin^2(\theta/3) - 8$$

has exactly one zero on $\mathbb{R}$.

Problem 6. Find all real zeros of the following polynomials.

(a) $f(x) = x^3 + 3x^2 - 4x - 12$ (Factor by Grouping)

(b) $g(x) = x^4 - 2x^2 - 15$ (Factor by Substitution $u = x^2$)

Problem 7. Find all real zeros of the following polynomials.

(a) $p(x) = x^3 - 4x^2 - 11x + 30$ (Integer Zeros Theorem)

(b) $q(x) = 3x^3 + 11x^2 - 19x + 5$ (Rational Zeros Theorem)

Problem 8 (Thomas §3.6 # 46). Consider the equation

$$(x^2 + y^2)^2 = (x - y)^2.$$

Problem 9 (Thomas §3.8 # 27). A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

Problem 10 (Thomas §4.1 #4). Let

$$f(x) = \frac{x+1}{x^2 + 2x + 2}.$$

Find all local extreme values of the function f, and where they occur.

AP Calc AB H4.2a Tues Dec 17, 2019

MVT

#1 $f(x)=\sqrt{x-1}$ on $[1,3]$

$f'(x)=\dfrac{1}{2\sqrt{x-1}}$ $m=\dfrac{f(b)-f(a)}{b-a}=\dfrac{f(3)-f(1)}{2}=\dfrac{\sqrt{2}}{2}=\dfrac{1}{\sqrt{2}}$

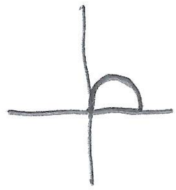
so $f'(c)=\dfrac{1}{2\sqrt{c-1}}=\dfrac{1}{\sqrt{2}} \Rightarrow 4(c-1)=2 \Rightarrow c-1=\frac{1}{2} \Rightarrow \boxed{c=\frac{3}{2}}$

#2 Satisfy MVT?

(a) $f(x)=x^{2/3}$ on $[-1,8]$: No, not diff at $x=0$

(b) $f(x)=x^{4/5}$ on $[0,1]$: Yes, it is cont on $[0,1]$ and diff on $(0,1)$
It isn't diff at an endpoint but that's OK.

(c) $f(x)=\sqrt{x(1-x)}$ on $[0,1]$: Yes, it is a semicircle.



$=\sqrt{x-x^2}=\sqrt{-(x^2-x+\frac{1}{4})+\frac{1}{4}}$

$=\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}$

Now $y^2=\frac{1}{4}-(x-\frac{1}{2})^2$ so $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$ so circle radius $\frac{1}{2}$ center $(\frac{1}{2},0)$

(d) $f(x)=\begin{cases}\frac{\sin x}{x} & \text{for } x\in[-\pi,0) \\ 0 & \text{for } x=0\end{cases}$ Yes, f is in fact continuous at $x=0$.

#3 Let

$f(x)=\begin{cases}3 & \text{for } x=0 \\ -x^2+3x+a & \text{for } x\in[0,1) \\ mx+b & \text{for } x\in[1,2]\end{cases}$

Since f is cont, $-x^2+3x+a\Big|_{x=0}=3$, so $a=3$. Differentiating at $x=1$ gives $-2x+3=m$ for $x=1$, so $m=1$.

Continuity at $x=1$ gives $-x^2+3x+3=x+b$, so $-1+3+3=1+b$, so $b=4$

$\boxed{a=3,\ m=1,\ b=4}$

#4 Show $f(x)=x^4+3x+1$ has exactly one zero on $[-2,-1]$.

Note $f(-2)=16-6+1=11$, and $f(-1)=1-3+1=-1$.

By IVT, f has a zero in $(-2,-1)$. Note $f'(x)=4x^3+3$

~~If~~ f has another zero, ~~say~~ f has 2 zeros x_1, x_2 w/ $x_1<x_2$,

then $f'(c)=0$ for some $c\in(x_1,x_2)$. But $4x^3+3<0$ on (x_1,x_2),

So $f'\neq 0$ anywhere on $(-2,-1)$, so there are not 2 zeros.

#5) Show that $r(\theta) = \theta + \sin^2(\theta/3) - 8$

Has exactly one zero on $\mathbb{R}$.

$\frac{dr}{d\theta} = 1 + 2\sin(\theta/3)(\cos\theta/3)\frac{1}{3} = 1 + \frac{1}{3}\sin\left(\frac{2\theta}{3}\right) \geq 0.$

If r has 2 zero, Rolle's Thm would imply $r' = 0$; by $r' > 0$.

So r does not have 2 zeros.

But $r(0) = -8$ and $r(\theta) = \theta + \sin^2(3) - 8 > 0$, so $r = 0$ somewhere by IVT

#6) Find all real zeros of:

ⓐ $f(x) = x^3 + 3x^2 - 4x - 12$

$= x^2(x+3) - 4(x+3)$

$= (x-2)(x+2)(x+3)$

ⓐ $f(x) = x^3 + 3x^2 - 4x + 12$

$f'(x) = 3x^2 + 6x - 4$

$f'=0 \Rightarrow x = -6 \pm \sqrt{36+4}$

ⓑ $g(x) = x^4 - 2x^2 - 15$

$= (x^2-5)(x^2+3)$ zeros are $\pm\sqrt{5}$.

#7) ⓐ $p(x) = x^3 - 4x^2 - 11x + 30$

was −

does not

$$\begin{array}{r|rrrr} 3 & 1 & -4 & -11 & -30 \\ & & 5 & 5 & \\ \hline & 1 & 1 & -6 & \end{array}$$

ⓑ $q(x) = 3x^3 + 11x^2 - 19x + 5$

$= (x-1)(3x^2 + 14x - 5)$

$= (x-1)(3x-1)(x+5)$ $x = 1, \frac{1}{3}, -5$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -19 & 5 \\ & & 3 & 14 & -5 \\ \hline & 3 & 14 & -5 & \boxed{0} \end{array}$$

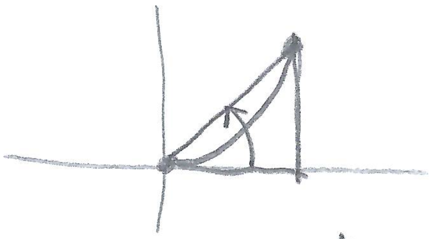
#8) $(x^2+y^2)^2 = (x-y)$ slope at $(1,0)$ and $(1,-1)$

$2(x^2+y^2)(2x + 2y\frac{dy}{dx}) = 1 - \frac{dy}{dx}$

at $(1,0)$, $2(1)(2)\frac{dy}{dx} = 1 - \frac{dy}{dx} \Rightarrow 5y' = 1 \Rightarrow y' = \frac{1}{5}$

at $(1,-1)$ $2(2)(2-2) = 1 - \frac{dy}{dx} \Rightarrow y' = 1$

#9 $y = x^2$



$\tan\theta = \frac{x^2}{x} = x$ so $\cancel{\frac{d}{dt}}$ $\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt} = 10$

$x=3 \Rightarrow \tan\theta = 3 \Rightarrow \tan^2\theta = 9 \Rightarrow \sec^2\theta = 10 \Rightarrow \frac{d\theta}{dt} = \frac{10}{10} = \mathbf{1}\ \frac{\text{rad}}{\text{sec}}$

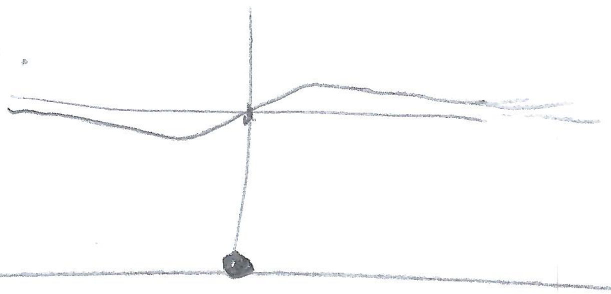
#10 $f(x) = \frac{x+1}{x^2+2x+2} = \frac{x+1}{1+(x+1)^2}$

shift of $\frac{x}{1+x^2}$

$$f'(x) = \frac{(x^2+2x+2) - (x+1)(2x+2)}{(x^2+2x+2)^2}$$

$$= \frac{x^2+2x+2-2x^2-4x-2}{(x^2+2x+2)^2} = \frac{-x^2-2x}{(x^2+2x+2)^2}$$

$f'=0 \Rightarrow x^2+2x=0 \Rightarrow x=0$ or $x=-2$.



$f(0) = \frac{1}{2}$

$f(-2) = \frac{-1}{4-4+2} = -\frac{1}{2}$